Discrete Mathematics 310 (2010) 2776–2789

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Hamilton cycle decompositions of the tensor products of complete bipartite graphs and complete multipartite graphs

R.S. Manikandan [1], P. Paulraja *

Department of Mathematics, Annamalai University, Annamalainagar 608 002, India

ARTICLE INFO

Article history:
Received 21 January 2009
Received in revised form 17 May 2010
Accepted 30 May 2010
Available online 9 July 2010

Keywords:
Tensor product
Wreath product
Hamilton cycle decomposition

ABSTRACT

In this paper, it is shown that the tensor product of the complete bipartite graph, $K_{r,r}$, $r \geq 2$, and the regular complete multipartite graph, $K_m * \overline{K}_n$, $m \geq 3$, is Hamilton cycle decomposable.

1. Introduction

A k-regular graph G has a *Hamilton cycle decomposition* if its edge set can be partitioned into $\frac{k}{2}$ Hamilton cycles when k is even, or into $(k-1)/2$ Hamilton cycles plus a 1-factor (or a perfect matching) when k is odd. We write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ if $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$. The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$. C_m denotes the cycle of length m.

For two simple graphs G and H their *wreath product*, denoted by $G * H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Similarly, $G \times H$, the *tensor product* of the graphs G and H has vertex set $V(G) \times V(H)$ in which two vertices $(g_1, h_1)$and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. The tensor product is known to be commutative and distributive over an edge-disjoint union of subgraphs, that is, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$.

We shall use the following notation throughout the paper. Let G and H be simple graphs with $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$. Then $V(G \times H) = V(G) \times V(H)$. For our convenience, we write $V(G) \times V(H) = \bigcup_{i=1}^{m} X_i$, where X_i stands for $\{x_i\} \times V(H)$. Further, in the sequel, we shall denote the set of vertices of X_i as $\{x_j^i \mid 1 \leq j \leq n\}$, where x_j^i stands for the vertex (x_i, u_j). X_i, $1 \leq i \leq m$, is called the ith *layer* of $G \times H$. We shall call $G \times H$ an m-partite graph with partite sets $X_1, X_2, \ldots, X_m$. (We can also consider $G \times H$ as an n-partite graph with partite sets $U_i = V(G) \times \{u_i\}$, $1 \leq i \leq n$.)

Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. If x_iy_j is an edge of G, then x_iy_j is called an edge of *distance* $j-i$ *from* X *to* Y if $i \leq j$, or $n-(i-j)$ if $i > j$. The same edge is said to be of *distance* $i-j$ *from* Y *to* X if $i \geq j$ or $n-(j-i)$, if $i < j$. If G contains the set of edges $F_i(X, Y) = \{x_jy_{i+j} \mid 1 \leq j \leq n\}$, $0 \leq i \leq n-1$, where the addition in the subscript is taken modulo n with residues $1, 2, \ldots, n$, then we say that G has a 1-*factor of distance* i *from* X *to* Y. Note that $F_i(X, Y) = F_{n-i}(Y, X) = \{y_kx_{k+n-i} \mid 1 \leq k \leq n\}$, $0 \leq i \leq n-1$. Clearly, if $G = K_{n,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y)$.

* Corresponding author.
E-mail address: pprajaau@sify.com (P. Paulraja).
[1] Current address: Department of Mathematics, Bharathidasan University College(W), Orathanadu-614 625, Tamil Nadu, India.

0012-365X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2010.05.034

Let k be a positive integer and let L be a subset of $\{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$. A *circulant* $X = X(k; L)$ is a graph with vertex set $V(X) = \{u_1, u_2, \ldots, u_k\}$ and the edge set $E(X) = \{u_i u_{i+l} \mid i \in \{1, 2, \ldots, k\}, l \in L$, where addition in the subscript of u is taken modulo k with residues $1, 2, \ldots, k\}$. The edge $u_i u_{i+l}, l \in L$, is said to be of distance l, and L is called the edge distance set of the circulant X. It is clear that if $gcd(k, l) = 1$, then the circulant $X(k; \{l\})$ is a Hamilton cycle. We shall name the graph isomorphic to $X(2r; \{1, r\})$ as W_{2r}.

For a digraph D, by $A(D)$ we mean the arc set of D. Definitions which are not given here can be found in [4] or [7].

In this paper, we study the Hamilton cycle decomposition of $K_{r,r} \times (K_m * \overline{K}_n), r \geq 2, m \geq 3$. The problem of finding Hamilton cycle decompositions of product graphs is not new. Hamilton cycle decompositions of various product graphs have been studied by many authors; see, for example, [1,6,8,9,11–14]. It has been conjectured [6] that if both G and H are Hamilton cycle decomposable graphs, then $G \Box H$ is Hamilton cycle decomposable, where $\Box$ denotes the cartesian product of graphs [1]. This conjecture has been verified to be true for a large class of graphs [14]. Baranyai and Szasz [5] proved that if both G and H are even regular Hamilton cycle decomposable graphs, then $G * H$ is Hamilton cycle decomposable. In [13], Ng has obtained a partial solution to the following conjecture of Alspach et al. [1]: If D_1 and D_2 are directed Hamilton cycle decomposable digraphs, then $D_1 * D_2$ is directed Hamilton cycle decomposable. Jha [9] has advanced the following conjecture: if both G and H are Hamilton cycle decomposable graphs and $G \times H$ is connected, then $G \times H$ is Hamilton cycle decomposable. But this conjecture is disproved in [3]. Because of this, finding Hamilton cycle decompositions of the tensor product of Hamilton cycle decomposable graphs is considered to be difficult. In [2] it has been proved that $K_r \times K_s$ is Hamilton cycle decomposable. In [11] $K_{r,r} \times K_m$ is shown to be Hamilton cycle decomposable and in [12] it is shown that the tensor product of two regular complete multipartite graphs is Hamilton cycle decomposable. In this paper, we prove the following:

Theorem 1.1. *If $r \geq 2$ and $m \geq 3$, then $K_{r,r} \times (K_m * \overline{K}_n)$ has a Hamilton cycle decomposition.*

2. Proof of the main theorem

First we prove a few lemmas; using these lemmas we prove the main result of this paper.

Throughout the following lemma and its proof we assume that the subscripts of X_i's and the superscripts of x_i^j's are taken modulo m with residues $1, 2, \ldots, m$ and the subscripts of x_i^j's are taken modulo n with residues $1, 2, \ldots, n$; the addition in the distance of 1-factors is taken modulo n with residues $0, 1, 2, \ldots, n-1$, and $0 \leq \alpha, \beta_i, \gamma_i \leq n-1, 3 \leq i \leq m$.

Lemma 2.1. *Let the vertex set of the m-partite graph $C_m \times K_n, m \geq 3$ be $X_i = \{x_1^i, x_2^i, \ldots, x_n^i\}, 1 \leq i \leq m$. Let G be a spanning subgraph of $C_m \times K_n$ containing the edges of $F_1 \cup F_2$, where $F_1 = F_\alpha(X_1, X_2) \cup F_\alpha(X_2, X_3) \cup \left(\bigcup_{j=3}^m F_{\beta_j}(X_j, X_{j+1})\right)$, $F_2 = F_{\alpha+1}(X_1, X_2) \cup F_{\alpha+1}(X_2, X_3) \cup \left(\bigcup_{j=3}^m F_{\gamma_j}(X_j, X_{j+1})\right)$, $\beta_j \neq \gamma_j$, and $F_k(X_i, X_j)$ denotes the 1-factor of distance k from X_i to X_j; then*

(a) *if $gcd(2\alpha + \sum_{j=3}^m \beta_j, n) = 1$ and $gcd(2\alpha + 2 + \sum_{j=3}^m \gamma_j, n) = 2$, then $F_1 \cup F_2$ can be decomposed into two Hamilton cycles of G;*
(b) *if $gcd(2\alpha + \sum_{j=3}^m \beta_j, n) = 2$ and $gcd(2\alpha + 2 + \sum_{j=3}^m \gamma_j, n) = 1$, then $F_1 \cup F_2$ can be decomposed into two Hamilton cycles of G;*
(c) *if $gcd(2\alpha + \sum_{j=3}^m \beta_j, n) = 2$ and $gcd(2\alpha + 2 + \sum_{j=3}^m \gamma_j, n) = 2$, then $F_1 \cup F_2$ can be decomposed into two Hamilton cycles of G;*
(d) *if $n \geq 8$, $gcd(2\alpha + \sum_{j=3}^m \beta_j, n) = 4$ and $gcd(2\alpha + 2 + \sum_{j=3}^m \gamma_j, n) = 4$, then $F_1 \cup F_2$ can be decomposed into two Hamilton cycles of G.*

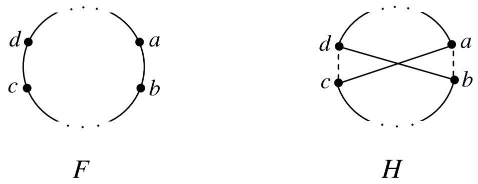
Proof. Proof of (a). Clearly, F_1 is a Hamilton cycle and F_2 is a 2-factor of G consisting of two cycles C' and C'' of equal length.

The vertices $x_1^1, x_3^1, x_5^1, \ldots, x_{n-1}^1$ are contained in a single cycle of F_2, say, C', and the vertices $x_2^1, x_4^1, x_6^1, \ldots, x_n^1$ are contained in the other cycle C'' of F_2. Now we decompose $F_1 \cup F_2$ into two Hamilton cycles H' and H'' of G as follows: $H' = \left(F_1 - \{x_1^1 x_{\alpha+1}^2, x_{\alpha+2}^2 x_{2\alpha+2}^3\}\right) \cup \{x_1^1 x_{\alpha+2}^2, x_{\alpha+1}^2 x_{2\alpha+2}^3\}$ and $H'' = \left(F_2 - \{x_1^1 x_{\alpha+2}^2, x_{\alpha+1}^2 x_{2\alpha+2}^3\}\right) \cup \{x_1^1 x_{\alpha+1}^2, x_{\alpha+2}^2 x_{2\alpha+2}^3\}$. The fact that H' is a Hamilton cycle of G can be seen by letting $a = x_1^1, b = x_{\alpha+1}^2, c = x_{\alpha+2}^2, d = x_{2\alpha+2}^3, F = F_1$ and $H = H'$ in the graphs of Fig. 1.

The fact that H'' is a Hamilton cycle of G can be seen by letting $a = x_1^1, b = x_{\alpha+2}^2, c = x_{\alpha+1}^2, d = x_{2\alpha+2}^3, F = F_2$ and $H = H''$ in the graphs of Fig. 2.

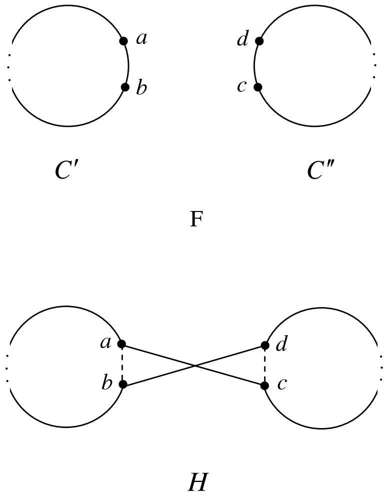
Proof of (b). Clearly, F_2 is a Hamilton cycle and F_1 is a 2-factor of G consisting of two cycles C' and C'' of equal length. The vertices $x_1^1, x_3^1, x_5^1, \ldots, x_{n-1}^1$ are contained in a single cycle of F_1, say, C', and the vertices $x_2^1, x_4^1, x_6^1, \ldots, x_n^1$ are contained in the other cycle C'' of F_1. Now we decompose $F_1 \cup F_2$ into two Hamilton cycles H' and H'' of G as follows:
$H' = \left(F_1 - \{x_1^1 x_{\alpha+1}^2, x_{\alpha+2}^2 x_{2\alpha+2}^3\}\right) \cup \{x_1^1 x_{\alpha+2}^2, x_{\alpha+1}^2 x_{2\alpha+2}^3\}$ and $H'' = \left(F_2 - \{x_1^1 x_{\alpha+2}^2, x_{\alpha+1}^2 x_{2\alpha+2}^3\}\right) \cup \{x_1^1 x_{\alpha+1}^2, x_{\alpha+2}^2 x_{2\alpha+2}^3\}$. The fact that H'' is a Hamilton cycle of G can be seen by letting $a = x_1^1, b = x_{\alpha+2}^2, c = x_{\alpha+1}^2, d = x_{2\alpha+2}^3, F = F_2$ and $H = H''$ in the graphs of Fig. 1. The fact that H' is a Hamilton cycle of G can be seen by letting $a = x_1^1, b = x_{\alpha+1}^2, c = x_{\alpha+2}^2, d = x_{2\alpha+2}^3, F = F_1$ and $H = H'$ in the graphs of Fig. 2.

Proof of (c). Clearly each $F_i, i = 1, 2$, is a 2-factor of G consisting of two cycles C_i' and C_i'' of equal length. The vertices $x_1^1, x_3^1, x_5^1, \ldots, x_{n-1}^1$ are contained in a single cycle of F_i, say, C_i' and the vertices $x_2^1, x_4^1, x_6^1, \ldots, x_n^1$ are contained



Broken edges represent the edges we have deleted from F for the construction of H.

Fig. 1.



Broken edges represent the edges we have deleted from F for the construction of H.

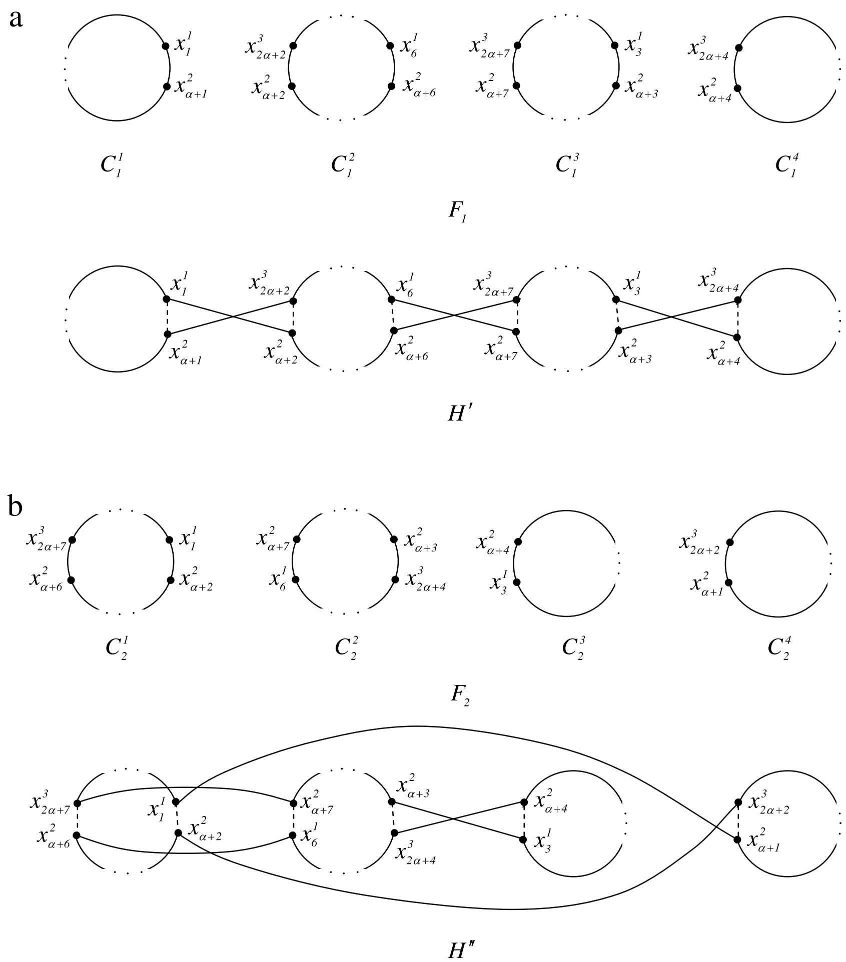
Fig. 2.

in the other cycle C_i'' of F_i. Now we decompose $F_1 \cup F_2$ into two Hamilton cycles H' and H'' of G as follows: $H' = \left(F_1 - \{x_1^1x_{\alpha+1}^2, x_{\alpha+2}^2x_{2\alpha+2}^3\}\right) \cup \{x_1^1x_{\alpha+2}^2, x_{\alpha+1}^2x_{2\alpha+2}^3\}$ and $H'' = \left(F_2 - \{x_1^1x_{\alpha+2}^2, x_{\alpha+1}^2x_{2\alpha+2}^3\}\right) \cup \{x_1^1x_{\alpha+1}^2, x_{\alpha+2}^2x_{2\alpha+2}^3\}$.
The fact that H' is a Hamilton cycle of G can be seen by letting $a = x_1^1, b = x_{\alpha+1}^2, c = x_{\alpha+2}^2, d = x_{2\alpha+2}^3, C' = C_1', C'' = C_1'', F = F_1$ and $H = H'$ in the graphs of Fig. 2. The fact that H'' is a Hamilton cycle of G can be seen by letting $a = x_1^1, b = x_{\alpha+2}^2, c = x_{\alpha+1}^2, d = x_{2\alpha+2}^3, C' = C_2', C'' = C_2'', F = F_2$ and $H = H''$ in the graphs of Fig. 2.

Proof of (d). Clearly, each $F_i, i = 1, 2$, is a 2-factor of G consisting of four cycles, say, C_i^1, C_i^2, C_i^3 and C_i^4, of equal length. By the choice of F_i, we suppose $x_{k+4l}^1 \in C_i^k, 1 \le k \le 4, 0 \le l \le \frac{n}{4} - 1$. Now we decompose $F_1 \cup F_2$ into two Hamilton cycles H' and H'' as follows: $H' = \left(F_1 - \{x_1^1x_{\alpha+1}^2, x_{\alpha+2}^2x_{2\alpha+2}^3, x_3^1x_{\alpha+3}^2, x_{\alpha+4}^2x_{2\alpha+4}^3, x_6^1x_{\alpha+6}^2, x_{\alpha+7}^2x_{2\alpha+7}^3\}\right) \cup \{x_1^1x_{\alpha+2}^2, x_{\alpha+1}^2x_{2\alpha+2}^3, x_3^1x_{\alpha+4}^2, x_{\alpha+3}^2x_{2\alpha+4}^3, x_6^1x_{\alpha+7}^2, x_{\alpha+6}^2x_{2\alpha+7}^3\}$ and $H'' = \left(F_2 - \{x_1^1x_{\alpha+2}^2, x_{\alpha+1}^2x_{2\alpha+2}^3, x_3^1x_{\alpha+4}^2, x_{\alpha+3}^2x_{2\alpha+4}^3, x_6^1x_{\alpha+7}^2, x_{\alpha+6}^2x_{2\alpha+7}^3\}\right) \cup \{x_1^1x_{\alpha+1}^2, x_{\alpha+2}^2x_{2\alpha+2}^3, x_3^1x_{\alpha+3}^2, x_{\alpha+4}^2x_{2\alpha+4}^3, x_6^1x_{\alpha+6}^2, x_{\alpha+7}^2x_{2\alpha+7}^3\}$. Indeed, H' and H'' are Hamilton cycles of G; see Fig. 3. □

Remark 2.2. From the construction of the Hamilton cycles H' and H'' of the above lemma, it is clear that for every vertex in the ith layer, $1 \le i \le m$, except in the second layer, the preceding and succeeding vertices of any vertex in the Hamilton cycles H' and H'' are in X_{i-1} and X_{i+1} and for some vertices in the second layer both the preceding and succeeding vertices are in X_1 or X_3. Consequently, in the subgraph obtained by the intersection of H' (resp. H'') with the subgraph induced by the first three layers of G all the vertices in X_1 and X_3 are of degree 1 in the subgraph. Further, the intersection of H' (resp. H'') with the subgraph of G induced by $X_i \cup X_{i+1}$, $3 \le i \le m$, is a 1-factor (of the induced subgraph). This fact will be used later. □

For rest of the results, except Theorem 1.1, proved in this paper we assume the following: let $m \ge 4$ be even and let $\{u_1, u_2, \ldots, u_{mn}\}$ be the vertex set of $K_m * \overline{K}_n$. Place the vertices of $K_m * \overline{K}_n$ in the circular order $\{u_1, u_2, \ldots, u_{mn}\}$. Let the



Broken edges of Figure 3 (a) (resp. Figure 3 (b)) represent the edges we have deleted from F_1 (resp. F_2) for the construction of H' (resp. H'')

Fig. 3.

ith partite set of $K_m * \overline{K}_n$ be $U_i = \{u_{i+jm} \mid 0 \leq j \leq n-1\}$, $1 \leq i \leq m$. Thus $K_m * \overline{K}_n$ is isomorphic to the circulant $X(mn; \{1, 2, \ldots, mn/2\} - \{im \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\})$. When considering the graph $W_{2r} \times (K_m * \overline{K}_n)$, we assume that the graph $K_m * \overline{K}_n$ is given in its circulant form. Hence the edge x_ix_j in W_{2r} and the edge u_ku_{k+l} in $K_m * \overline{K}_n$ give rise to the two edges $x_k^ix_{k+l}^j$ and $x_{k+l}^ix_k^j$ in $W_{2r} \times (K_m * \overline{K}_n)$. Hence corresponding to the edge x_ix_j in W_{2r}, the edge set of the subgraph induced by $X_i \cup X_j$ in $W_{2r} \times (K_m * \overline{K}_n)$ is $\bigcup_{l \in L} F_l(X_i, X_j)$, where $L = \{1, 2, \ldots, mn\} - \{km \mid 1 \leq k \leq n\}$, since the edge x_ix_j in W_{2r} and the set of edges of distance $l \in \{1, 2, \ldots, mn/2\} - \{im \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ in $K_m * \overline{K}_n$ give rise to the edges of $F_l(X_i, X_j)$ and $F_{mn-l}(X_i, X_j)$.

Lemma 2.3. *If $r \geq 3$ is odd, $m \geq 4$ and $n \geq 2$ are even, then $W_{2r} \times (K_m * \overline{K}_n)$ is Hamilton cycle decomposable.*

Proof. Throughout the proof of this lemma the subscripts of x_i's and X_i's and the superscripts of x_j^i's are taken modulo $2r$ with residues $1, 2, \ldots, 2r$ and the subscripts of x_j^i's are taken modulo mn with residues $1, 2, \ldots, mn$. Let the vertex set of W_{2r} be $\{x_1, x_2, x_3, \ldots, x_{2r}\}$. Then its edge set can be described as $\{x_ix_{i+1} \mid 1 \leq i \leq 2r\} \cup \{x_ix_{i+r} \mid 1 \leq i \leq r\}$. Let the partite sets of the $2r$-partite graph $W_{2r} \times (K_m * \overline{K}_n)$ be $X_i = \{x_1^i, x_2^i, x_3^i, \ldots, x_{mn}^i\}$, $1 \leq i \leq 2r$. Then the edge set of $W_{2r} \times (K_m * \overline{K}_n)$ can be described as $\left(\bigcup_{i=1}^{2r}\left(\bigcup_{j \in L} F_j(X_i, X_{i+1})\right)\right) \cup \left(\bigcup_{i=1}^{r}\left(\bigcup_{j \in L} F_j(X_i, X_{i+r})\right)\right)$, where $L = \{1, 2, \ldots, mn\} - \{km \mid 1 \leq k \leq n\}$.

First we decompose $W_{2r} \times (K_m * \overline{K}_n)$ into three mutually isomorphic connected spanning subgraphs, G_t, $t = 1, 2, 3$, and then we decompose G_t into Hamilton cycles. This isomorphic decomposition is achieved by obtaining the following directed Hamilton cycle decomposition of W_{2r}^*, the digraph arises out of W_{2r} by replacing each one of its edges by

Table 1

F_1	$0 \le j \le \frac{n}{2}-1, 2 \le i \le m-1,$ F_{jm+i}	$1 \le j \le \frac{n}{2}-1,$ F_{jm+1}
1	$jm+i$	$jm+1$
1	$jm+i$	$jm+1$
$\frac{mn}{2}-1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-1$
$\frac{mn}{2}-1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-1$
1	$jm+i$	$jm+1$
1	$\frac{mn}{2}-jm-i+1$	$\frac{mn}{2}-jm+1$
1	$jm+i$	$jm+1$
1	$jm+i$	$jm+1$
$\frac{mn}{2}-1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-1$
$\frac{mn}{2}-1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-1$
.	.	.
.	.	.
.	.	.
1	$jm+i$	$jm+1$
1	$jm+i$	$jm+1$
$\frac{mn}{2}-1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-1$
$\frac{mn}{2}-1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-1$

a symmetric pair of arcs. The required directed Hamilton cycle decomposition $\{\vec{H}_1, \vec{H}_2, \vec{H}_3\}$ of W_{2r}^* is given by $\vec{H}_1 = \bigcup_{i=1}^{r}\{(x_{r+2i-1}, x_{2i-1}), (x_{2i-1}, x_{2i})\}$, $\vec{H}_2 = \bigcup_{i=1}^{r}\{(x_{2i-2}, x_{2i-1}), (x_{2i-1}, x_{r+2i-1})\}$and $\vec{H}_3 = \bigcup_{i=1}^{2r}\{(x_i, x_{i-1})\}$.

Clearly, $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ are arc-disjoint directed Hamilton cycles of W_{2r}^*. We define G_t, $t = 1, 2, 3$, as follows:
Let $L' = \{1, 2, \dots, mn/2\} - \{km \mid 1 \le k \le n/2\}$.

Then $G_t = \bigcup_{(x_i,x_j)\in A(\vec{H}_t)} \left(\bigcup_{l\in L'} F_l(X_i, X_j)\right)$, $t = 1, 2, 3$. Using the fact that $F_l(X_i, X_j) = F_{mn-l}(X_j, X_i)$, we can check that $W_{2r} \times (K_m * \overline{K}_n) = \bigoplus_{t=1}^{3} G_t$. Clearly,$G_t$, $t = 1, 2, 3$, is isomorphic to G, where $G = \bigcup_{i=1}^{2r}\{\bigcup_{l\in L'} F_l(X_i, X_{i+1})\}$. Thus to prove the existence of a Hamilton cycle decomposition of G_t, $t = 1, 2, 3$, it is enough to decompose G into Hamilton cycles. We divide the proof into two cases.

Case 1. $mn \equiv 0 \pmod 8$.
We initially obtain a 2-factorization of G and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose each of them into two Hamilton cycles of G. We obtain a 2-factorization of G by describing the 2-factors F_i, $i \in (\{1, 2, \dots, mn/2\} - \{km \mid 1 \le k \le n/2\})$, as given in Table 1.

In Table 1, the 1 (resp. $jm + i$, $jm + 1$) in the first row denotes the distance of the 1-factor from X_1 to X_2 that we have chosen for the construction of the 2-factor F_1 (resp. F_{jm+i}, F_{jm+1}). Similarly, the 1 (resp. $jm + i$, $jm + 1$) in the second row denotes the distance of the 1-factor from X_2 to X_3 that we have chosen for the construction of F_1 (resp. F_{jm+i}, F_{jm+1}) and so on. That is, an s in the pth row of the table denotes the distance of the 1-factor from X_p to X_{p+1} that we have chosen for the construction of F_1 or F_{jm+i} or F_{jm+1} according to whether s is in column 1 or 2 or 3. Further, in the table every successive four rows, except the first six rows, are identical, in order, and the four entries in each of the three columns described by these four rows add up to a multiple of mn.

Clearly, all the 2-factors defined above, except the $n/2$ 2-factors, F_1 and F_{jm+1}, $1 \le j \le \frac{n}{2} - 1$, are Hamilton cycles. These $n/2$ 2-factors are combined with $n/2$ Hamilton cycles, namely, F_{jm+2}, $0 \le j \le \frac{n}{2} - 1$, in pairs, to obtain a Hamilton cycle decomposition of G as follows: $F_{jm+1} \cup F_{jm+2}$, $0 \le j \le \frac{n}{2} - 1$, can be decomposed into two Hamilton cycles, by Lemma 2.1.

Case 2. $mn \equiv 4 \pmod 8$.
As in the above case, we obtain a 2-factorization of G and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of G. We obtain a 2-factorization of G by describing the 2-factors F_i, $i \in (\{1, 2, \dots, mn/2\} - \{km \mid 1 \le k \le n/2\})$ of G as given in Table 2. In Table 2, every successive four rows, except the first six rows, are identical, in order, and the four entries in each of the six columns described by these four rows add up to a multiple of mn.

Clearly, all the 2-factors of G defined above, except the $n/2$ 2-factors, F_1 and F_{jm+3}, $1 \le j \le \frac{n}{2} - 1$, are Hamilton cycles. Now we combine these $n/2$ 2-factors with the $n/2$ Hamilton cycles, F_2 and F_{jm+2}, $1 \le j \le \frac{n}{2} - 1$, in pairs, to obtain a Hamilton cycle decomposition of G as follows: $F_1 \cup F_2$ and $F_{jm+2} \cup F_{jm+3}$, $1 \le j \le \frac{n}{2} - 1$, can be decomposed into two Hamilton cycles, by Lemma 2.1. This completes the proof. □

For the rest of the paper, except the proof of Theorem 1.1, we assume the following:

1. $m \ge 4$ is even, $n \ge 3$ is odd.

Table 2

F_1	$2 \le i \le m-1$, F_i	$1 \le j \le \frac{n}{2}-1$, F_{jm+1}	F_{jm+2}	F_{jm+3}	$1 \le j \le \frac{n}{2}-1, 4 \le i \le m-1$, F_{jm+i}
1	i	$jm+1$	$jm+2$	$jm+3$	$jm+i$
1	i	$jm+1$	$jm+2$	$jm+3$	$jm+i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-1$	$\frac{mn}{2}-jm-2$	$\frac{mn}{2}-jm-3$	$\frac{mn}{2}-jm-i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-1$	$\frac{mn}{2}-jm-2$	$\frac{mn}{2}-jm-3$	$\frac{mn}{2}-jm-i$
1	i	$jm+1$	$jm+2$	$jm+3$	$jm+i$
1	$\frac{mn}{2}-i+1$	$\frac{mn}{2}-jm-2$	$\frac{mn}{2}-jm-1$	$\frac{mn}{2}-jm+1$	$\frac{mn}{2}-jm-i+1$
1	i	$jm+1$	$jm+2$	$jm+3$	$jm+i$
1	i	$jm+1$	$jm+2$	$jm+3$	$jm+i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-1$	$\frac{mn}{2}-jm-2$	$\frac{mn}{2}-jm-3$	$\frac{mn}{2}-jm-i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-1$	$\frac{mn}{2}-jm-2$	$\frac{mn}{2}-jm-3$	$\frac{mn}{2}-jm-i$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
1	i	$jm+1$	$jm+2$	$jm+3$	$jm+i$
1	i	$jm+1$	$jm+2$	$jm+3$	$jm+i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-1$	$\frac{mn}{2}-jm-2$	$\frac{mn}{2}-jm-3$	$\frac{mn}{2}-jm-i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-1$	$\frac{mn}{2}-jm-2$	$\frac{mn}{2}-jm-3$	$\frac{mn}{2}-jm-i$

2. A denotes the set $\{1, 2, 3, \dots, \frac{mn}{2}-1\} - \{im \mid 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$, B denotes the set $\{\frac{mn}{2}+1, \frac{mn}{2}+2, \frac{mn}{2}+3, \dots, mn-1\} - \{im \mid \lfloor \frac{n}{2} \rfloor + 1 \le i \le n-1\}$, $C = \{mn/2\}$, $D = B \cup C$ and $E = A \cup D$.
3. Let the vertex set of W_6 be $\{y_1, y_2, \dots, y_6\}$. For odd $r \ge 5$, let the vertex set of W_{2r} be $\{x_1, x_2, x_3, \dots, x_{2r}\}$; consequently, the 6-partite graph $W_6 \times (K_m * \overline{K}_n)$ has the vertex set $Y_i = \{y^i_1, y^i_2, y^i_3, \dots, y^i_{mn}\}$, $1 \le i \le 6$, and the vertex set of $W_{2r} \times (K_m * \overline{K}_n)$ is $X_i = \{x^i_1, x^i_2, x^i_3, \dots, x^i_{mn}\}$, $1 \le i \le 2r$. (For the later part of the paper, we need to have two different kinds of vertex sets of $W_{2r} \times (K_m * \overline{K}_n)$ when $r = 3$ and $r \ge 5$.)
4. The subscripts of y_i's and Y_i's and the superscripts of y^j_i's are taken modulo 6 with residues $1, 2, \dots, 6$. The subscripts of x_i's and X_i's and the superscripts of x^j_i's are taken modulo $2r$ with residues $1, 2, \dots, 2r$ $(r \ge 5)$ and the subscripts of x^j_i's and y^j_i's are taken modulo mn with residues $1, 2, \dots, mn$.
5. Clearly, the edge set of W_6 is $\{y_i y_{i+1} \mid 1 \le i \le 6\} \cup \{y_i y_{i+3} \mid 1 \le i \le 3\}$ and, for odd $r \ge 5$, the edge set of W_{2r} is $\{x_i x_{i+1} \mid 1 \le i \le 2r\} \cup \{x_i x_{i+r} \mid 1 \le i \le r\}$.
6. The edge set of the 6-partite graph $W_6 \times (K_m * \overline{K}_n)$ is given by $\left(\bigcup_{i=1}^{6} \left(\bigcup_{j\in E} F_j(Y_i, Y_{i+1})\right)\right) \cup \left(\bigcup_{i=1}^{3} \left(\bigcup_{j\in E} F_j(Y_i, Y_{i+3})\right)\right)$ and the edge set of the $2r$-partite graph $W_{2r} \times (K_m * \overline{K}_n)$ is given by $\left(\bigcup_{i=1}^{2r} \left(\bigcup_{j\in E} F_j(X_i, X_{i+1})\right)\right) \cup \left(\bigcup_{i=1}^{r} \left(\bigcup_{j\in E} F_j(X_i, X_{i+r})\right)\right)$.
7. G_1 and G_2 denote the spanning subgraphs of $W_6 \times (K_m * \overline{K}_n)$, where $G_1 = \bigcup_{i=1}^{6} \left(\bigcup_{j\in A} F_j(Y_i, Y_{i+1})\right)$ and $G_2 = \bigcup_{i=1}^{6} \left(\bigcup_{j\in D} F_j(Y_i, Y_{i+1})\right)$.

Lemma 2.4. *If $m \ge 4$ is even and $n \ge 3$ is odd, then $W_6 \times (K_m * \overline{K}_n) = G' \oplus G'' \oplus G''' \oplus F$, where $G' \cong G'' \cong G_1$, $G''' \cong G_2$ and F is a 1-factor of $W_6 \times (K_m * \overline{K}_n)$, where G_1 and G_2 are as described above.*

Proof. Let $G' = \left(\bigcup_{j\in A} F_j(Y_1, Y_2)\right) \cup \left(\bigcup_{j\in A} F_j(Y_2, Y_5)\right) \cup \left(\bigcup_{j\in A} F_j(Y_5, Y_6)\right) \cup \left(\bigcup_{j\in A} F_j(Y_6, Y_3)\right) \cup \left(\bigcup_{j\in A} F_j(Y_3, Y_4)\right) \cup \left(\bigcup_{j\in A} F_j(Y_4, Y_1)\right)$,
$G'' = \left(\bigcup_{j\in A} F_j(Y_2, Y_3)\right) \cup \left(\bigcup_{j\in A} F_j(Y_3, Y_6)\right) \cup \left(\bigcup_{j\in A} F_j(Y_6, Y_1)\right) \cup \left(\bigcup_{j\in A} F_j(Y_1, Y_4)\right) \cup \left(\bigcup_{j\in A} F_j(Y_4, Y_5)\right) \cup \left(\bigcup_{j\in A} F_j(Y_5, Y_2)\right)$,
$G''' = \bigcup_{i=1}^{6} \left(\bigcup_{j\in D} F_j(Y_i, Y_{i+1})\right)$ and let $F = \bigcup_{i=1}^{3} F_{mn/2}(Y_i, Y_{i+3})$. Clearly, $G' \cong G'' \cong G_1$, $G''' \cong G_2$ and F is a 1-factor of $W_6 \times (K_m * \overline{K}_n)$ and $W_6 \times (K_m * \overline{K}_n) = G' \oplus G'' \oplus G''' \oplus F$. □

From the above lemma, it is clear that to prove that $W_6 \times (K_m * \overline{K}_n)$ has a Hamilton cycle decomposition, it is enough to prove that the even regular graphs G_1 and G_2 have Hamilton cycle decompositions, which we prove below.

Lemma 2.5. *G_1 is Hamilton cycle decomposable.*

Proof. We prove this lemma in two cases.

Case 1. $m \equiv 4 \pmod 8$.
We prove this case in two subcases.

Table 3

F_1	F_2	F_3	F_5
1	2	3	5
1	2	3	5
5	2	3	1
5	2	3	1
3	1	5	2
2	1	5	3

Table 4

F_1	F_2	F_3	F_5	F_6	F_7	$2 \le j \le \lfloor \frac{n}{2} \rfloor - 1$,			F_{2n-1}
						F_{4j+1}	F_{4j+2}	F_{4j+3}	
1	2	3	5	6	7	$4j+1$	$4j+2$	$4j+3$	$2n-1$
1	2	3	5	6	7	$4j+1$	$4j+2$	$4j+3$	$2n-1$
$2n-1$	$2n-3$	$2n-4$	$2n-5$	$2n-7$	$2n-8$	$2n-4j-1$	$2n-4j-3$	$2n-4j-4$	1
$2n-1$	$2n-3$	$2n-4$	$2n-5$	$2n-7$	$2n-8$	$2n-4j-1$	$2n-4j-3$	$2n-4j-4$	1
1	2	6	3	5	7	$4j+2$	$4j+3$	$4j+1$	$2n-1$
1	$2n-1$	$2n-3$	$2n-4$	$2n-5$	$2n-7$	$2n-4j-1$	$2n-4j$	$2n-4j-3$	2

Table 5

F_1	$2 \le i \le \frac{m}{2}-2$,	$1 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, $1 \le i \le \frac{m}{2}-2$,	$0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$,			$0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, $\frac{m}{2}+2 \le i \le m-1$,	$1 \le i \le \frac{m}{2}-1$,
	F_i	F_{jm+i}	$F_{jm+\frac{m}{2}-1}$	$F_{jm+\frac{m}{2}}$	$F_{jm+\frac{m}{2}+1}$	F_{jm+i}	$F_{\lfloor \frac{n}{2} \rfloor m+i}$
1	i	$jm+i$	$jm+\frac{m}{2}-1$	$jm+\frac{m}{2}$	$jm+\frac{m}{2}+1$	$jm+i$	$\lfloor \frac{n}{2} \rfloor m+i$
1	i	$jm+i$	$jm+\frac{m}{2}-1$	$jm+\frac{m}{2}$	$jm+\frac{m}{2}+1$	$jm+i$	$\lfloor \frac{n}{2} \rfloor m+i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-\frac{m}{2}+1$	$\frac{mn}{2}-jm-\frac{m}{2}-1$	$\frac{mn}{2}-jm-\frac{m}{2}-2$	$\frac{mn}{2}-jm-i-1$	$\frac{mn}{2}-\lfloor \frac{n}{2} \rfloor m-i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-\frac{m}{2}+1$	$\frac{mn}{2}-jm-\frac{m}{2}-1$	$\frac{mn}{2}-jm-\frac{m}{2}-2$	$\frac{mn}{2}-jm-i-1$	$\frac{mn}{2}-\lfloor \frac{n}{2} \rfloor m-i$
1	i	$jm+i$	$jm+\frac{m}{2}$	$jm+\frac{m}{2}+1$	$jm+\frac{m}{2}-1$	$jm+i$	$\lfloor \frac{n}{2} \rfloor m+i$
1	$\frac{mn}{2}-i+1$	$\frac{mn}{2}-jm-i+1$	$\frac{mn}{2}-jm-\frac{m}{2}+1$	$\frac{mn}{2}-jm-\frac{m}{2}+2$	$\frac{mn}{2}-jm-\frac{m}{2}-1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-\lfloor \frac{n}{2} \rfloor m-i+1$

Subcase 1.1. $m = 4$.

If $n = 3$, then G_1 can be decomposed into four 2-factors F_1, F_2, F_3 and F_5 as shown in Table 3.

Clearly, F_1 and F_5 are Hamilton cycles of G_1, and F_2 and F_3 are 2-factors, each having two components, of G_1 since the greatest common divisor of the sum of the distances of the 1-factors from Y_i to Y_{i+1} (in the construction of F_2 and F_3) and mn (=12) is 2. But $F_2 \cup F_3$ can be decomposed into two Hamilton cycles, by Lemma 2.1.

Hence we assume that $n \ge 5$. First we obtain a 2-factorization of G_1 and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of G_1. A 2-factorization of G_1 is obtained by defining the 2-factors F_i, $i \in A$, as shown in Table 4.

The 2-factors F_1, F_2, F_6, F_7 and F_{4j+2}, F_{4j+3}, $2 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, are combined as $F_1 \cup F_2$, $F_6 \cup F_7$ and $F_{4j+2} \cup F_{4j+3}$, $2 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_1 from Table 4.

Subcase 1.2. $m \ge 12$.

As above, we obtain a 2-factorization of G_1 and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of G_1. A 2-factorization of G_1 is obtained by defining the 2-factors F_i, $i \in A$, as shown in Table 5.

The 2-factors $F_1, F_2, F_{jm+\frac{m}{2}}, F_{jm+\frac{m}{2}+1}$, $0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$ and $F_{jm+\frac{m}{2}+2i}, F_{jm+\frac{m}{2}+2i+1}$, $1 \le i \le \frac{m}{4} - 1$, $0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, are combined as $F_1 \cup F_2$, $F_{jm+\frac{m}{2}} \cup F_{jm+\frac{m}{2}+1}$, $0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, and $F_{jm+\frac{m}{2}+2i} \cup F_{jm+\frac{m}{2}+2i+1}$, $1 \le i \le \frac{m}{4} - 1$, $0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_1 from Table 5.

Case 2. $m \equiv 0, 2, 6 \pmod 8$.

As in the previous case, we obtain a 2-factorization of G_1. A 2-factorization of G_1 is obtained by defining the 2-factors F_i, $i \in A$, as shown in Table 6.

If $m \equiv 0 \pmod 8$, then the 2-factors $F_1, F_2, F_{jm+\frac{m}{2}}, F_{jm+\frac{m}{2}+1}$, $0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, and $F_{jm+\frac{m}{2}+2i}, F_{jm+\frac{m}{2}+2i+1}$, $1 \le i \le \frac{m}{4} - 1$, $0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, are combined as $F_1 \cup F_2$, $F_{jm+\frac{m}{2}} \cup F_{jm+\frac{m}{2}+1}$, $0 \le j \le \lfloor \frac{n}{2} \rfloor - 1$, and $F_{jm+\frac{m}{2}+2i} \cup F_{jm+\frac{m}{2}+2i+1}$, $1 \le$

Table 6

F_1	$2 \leq i \leq \frac{m}{2}-1$, F_i	$1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, $1 \leq i \leq \frac{m}{2}-1$, F_{jm+i}	$0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, $F_{jm+\frac{m}{2}}$	$0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, $\frac{m}{2}+1 \leq i \leq m-1$, F_{jm+i}	$1 \leq i \leq \frac{m}{2}-1$, $F_{\lfloor \frac{n}{2} \rfloor m+i}$
1	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$	$\lfloor \frac{n}{2} \rfloor m+i$
1	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$	$\lfloor \frac{n}{2} \rfloor m+i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-\frac{m}{2}-1$	$\frac{mn}{2}-jm-i-1$	$\frac{mn}{2}-\lfloor \frac{n}{2} \rfloor m-i$
$\frac{mn}{2}-1$	$\frac{mn}{2}-i$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-jm-\frac{m}{2}-1$	$\frac{mn}{2}-jm-i-1$	$\frac{mn}{2}-\lfloor \frac{n}{2} \rfloor m-i$
1	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$	$\lfloor \frac{n}{2} \rfloor m+i$
1	$\frac{mn}{2}-i+1$	$\frac{mn}{2}-jm-i+1$	$\frac{mn}{2}-jm-\frac{m}{2}+1$	$\frac{mn}{2}-jm-i$	$\frac{mn}{2}-\lfloor \frac{n}{2} \rfloor m-i+1$

Table 7

$F_{\frac{mn}{2}}$	$\frac{mn}{2}+1 \leq i \leq (\lfloor \frac{n}{2} \rfloor+1)m-1$,	$\lfloor \frac{n}{2} \rfloor+1 \leq j \leq n-1$, $1 \leq i \leq \frac{m}{2}-2$,	$\lfloor \frac{n}{2} \rfloor+1 \leq j \leq n-1$,			$\lfloor \frac{n}{2} \rfloor+1 \leq j \leq n-1$, $\frac{m}{2}+2 \leq i \leq m-1$,
	F_i	F_{jm+i}	$F_{jm+\frac{m}{2}-1}$	$F_{jm+\frac{m}{2}}$	$F_{jm+\frac{m}{2}+1}$	F_{jm+i}
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}-1$	$jm+\frac{m}{2}$	$jm+\frac{m}{2}+1$	$jm+i$
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}-1$	$jm+\frac{m}{2}$	$jm+\frac{m}{2}+1$	$jm+i$
$\frac{mn}{2}$	$\frac{3mn}{2}-i$	$\frac{3mn}{2}-jm-i+1$	$\frac{3mn}{2}-jm-\frac{m}{2}+2$	$\frac{3mn}{2}-jm-\frac{m}{2}+1$	$\frac{3mn}{2}-jm-\frac{m}{2}-1$	$\frac{3mn}{2}-jm-i$
$\frac{mn}{2}$	$\frac{3mn}{2}-i$	$\frac{3mn}{2}-jm-i+1$	$\frac{3mn}{2}-jm-\frac{m}{2}+2$	$\frac{3mn}{2}-jm-\frac{m}{2}+1$	$\frac{3mn}{2}-jm-\frac{m}{2}-1$	$\frac{3mn}{2}-jm-i$
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}+1$	$jm+\frac{m}{2}-1$	$jm+\frac{m}{2}$	$jm+i$
$mn-1$	$\frac{3mn}{2}-i-1$	$\frac{3mn}{2}-jm-i$	$\frac{3mn}{2}-jm-\frac{m}{2}+1$	$\frac{3mn}{2}-jm-\frac{m}{2}-2$	$\frac{3mn}{2}-jm-\frac{m}{2}-1$	$\frac{3mn}{2}-jm-i-1$

Table 8

$F_{\frac{mn}{2}}$	$\frac{mn}{2}+1 \leq i \leq (\lfloor \frac{n}{2} \rfloor+1)m-1$, F_i	$\lfloor \frac{n}{2} \rfloor+1 \leq j \leq n-1$, $1 \leq i \leq \frac{m}{2}-1$, F_{jm+i}	$\lfloor \frac{n}{2} \rfloor+1 \leq j \leq n-1$, $F_{jm+\frac{m}{2}}$	$\lfloor \frac{n}{2} \rfloor+1 \leq j \leq n-1$, $\frac{m}{2}+1 \leq i \leq m-1$, F_{jm+i}
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$
$\frac{mn}{2}$	$\frac{3mn}{2}-i$	$\frac{3mn}{2}-jm-i+1$	$\frac{3mn}{2}-jm-\frac{m}{2}+1$	$\frac{3mn}{2}-jm-i$
$\frac{mn}{2}$	$\frac{3mn}{2}-i$	$\frac{3mn}{2}-jm-i+1$	$\frac{3mn}{2}-jm-\frac{m}{2}+1$	$\frac{3mn}{2}-jm-i$
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$
$mn-1$	$\frac{3mn}{2}-i-1$	$\frac{3mn}{2}-jm-i$	$\frac{3mn}{2}-jm-\frac{m}{2}-1$	$\frac{3mn}{2}-jm-i-1$

$i \leq \frac{m}{4}-1$, $0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_1 from Table 6.

If $m \equiv 2, 6 \pmod 8$, then the 2-factors $F_1, F_2, F_{2i+1}, F_{2i+2}$, $1 \leq i \leq \frac{m-6}{4}$, $F_{jm+2i-1}, F_{jm+2i}$, $1 \leq i \leq \frac{m-2}{4}$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, $F_{jm+\frac{m}{2}}, F_{jm+\frac{m}{2}+1}$, $0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, and $F_{\lfloor \frac{n}{2} \rfloor m+2i-1}, F_{\lfloor \frac{n}{2} \rfloor m+2i}$, $1 \leq i \leq \frac{m-2}{4}$, are combined as $F_1 \cup F_2$, $F_{2i+1} \cup F_{2i+2}$, $1 \leq i \leq \frac{m-6}{4}$, $F_{jm+2i-1} \cup F_{jm+2i}$, $1 \leq i \leq \frac{m-2}{4}$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, $F_{jm+\frac{m}{2}} \cup F_{jm+\frac{m}{2}+1}$, $0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, and $F_{\lfloor \frac{n}{2} \rfloor m+2i-1} \cup F_{\lfloor \frac{n}{2} \rfloor m+2i}$, $1 \leq i \leq \frac{m-2}{4}$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_1 from Table 6. □

Lemma 2.6. *G_2 is Hamilton cycle decomposable.*

Proof. We prove this lemma in three cases.

Case 1. $m \equiv 4 \pmod 8$.
First we obtain a 2-factorization of G_2 and then we combine some of the 2-factors in the 2-factorization, in pairs, and decompose them into two Hamilton cycles of G_2. A 2-factorization of G_2 is obtained by defining the 2-factors F_i, $i \in D$, as shown in Table 7.
The 2-factors $F_{jm+2i-1}, F_{jm+2i}$, $1 \leq i \leq \frac{m-4}{4}$, $\lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-1$, and $F_{jm+\frac{m}{2}-1}, F_{jm+\frac{m}{2}}$, $\lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-1$, are combined as $F_{jm+2i-1} \cup F_{jm+2i}$, $1 \leq i \leq \frac{m-4}{4}$, $\lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-1$, and $F_{jm+\frac{m}{2}-1} \cup F_{jm+\frac{m}{2}}$, $\lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-1$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_2 from Table 7.

Case 2. $m \equiv 0 \pmod 8$.
As in the previous case, we obtain a 2-factorization of G_2. A 2-factorization of G_2 is obtained by defining the 2-factors F_i, $i \in D$, as shown in Table 8.

Table 9

$F_{\frac{mn}{2}}$	$\frac{mn}{2}+1 \le i \le (\lfloor \frac{n}{2} \rfloor + 1)m - 1$, F_i	$\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, $1 \le i \le \frac{m}{2}-1$, F_{jm+i}	$\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, $F_{jm+\frac{m}{2}}$	$\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, $\frac{m}{2}+1 \le i \le m-1$, F_{jm+i}
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$
$mn-1$	$\frac{3mn}{2}-i-1$	$\frac{3mn}{2}-jm-i$	$\frac{3mn}{2}-jm-\frac{m}{2}-1$	$\frac{3mn}{2}-jm-i-1$
$mn-1$	$\frac{3mn}{2}-i-1$	$\frac{3mn}{2}-jm-i$	$\frac{3mn}{2}-jm-\frac{m}{2}-1$	$\frac{3mn}{2}-jm-i-1$
$\frac{mn}{2}$	i	$jm+i$	$jm+\frac{m}{2}$	$jm+i$
$\frac{mn}{2}$	$\frac{3mn}{2}-i$	$\frac{3mn}{2}-jm-i+1$	$\frac{3mn}{2}-jm-\frac{m}{2}+1$	$\frac{3mn}{2}-jm-i$

The 2-factors $F_{jm+2i-1}, F_{jm+2i}$, $1 \le i \le \frac{m-4}{4}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, and $F_{jm+\frac{m}{2}-1}, F_{jm+\frac{m}{2}}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, are combined as $F_{jm+2i-1} \cup F_{jm+2i}$, $1 \le i \le \frac{m-4}{4}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, and $F_{jm+\frac{m}{2}-1} \cup F_{jm+\frac{m}{2}}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_2 from Table 8.

Case 3. $m \equiv 2, 6 \pmod 8$.
As above, first we obtain a 2-factorization of G_2. A 2-factorization of G_2 is obtained by defining the 2-factors F_i, $i \in D$, as shown in Table 9.
The 2-factors $F_{\frac{mn}{2}}, F_{\frac{mn}{2}+1}, F_{jm+2i-1}, F_{jm+2i}$, $1 \le i \le \frac{m-2}{4}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, and $F_{jm+\frac{m}{2}}, F_{jm+\frac{m}{2}+1}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, are combined as $F_{\frac{mn}{2}} \cup F_{\frac{mn}{2}+1}$, $F_{jm+2i-1} \cup F_{jm+2i}$, $1 \le i \le \frac{m-2}{4}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, and $F_{jm+\frac{m}{2}} \cup F_{jm+\frac{m}{2}+1}$, $\lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1$, and each one of them is decomposed into two Hamilton cycles, by Lemma 2.1. The rest of the 2-factors are readily seen to be Hamilton cycles of G_2 from Table 9. □

Remark 2.7. From the construction of every Hamilton cycle H of the Hamilton cycle decomposition of G_1 (resp. G_2) obtained in Lemma 2.5 (resp. Lemma 2.6), the vertices in Y_1 and Y_3 are of degree 1 in the subgraph $G_1[Y_1 \cup Y_2 \cup Y_3] \cap H$ (resp. $G_2[Y_1 \cup Y_2 \cup Y_3] \cap H$), obtained by taking the intersection of H with the subgraph induced by $Y_1 \cup Y_2 \cup Y_3$ in G_1 (resp. G_2). This is true as we use the Hamilton cycle decomposition described in Lemma 2.1; see Remark 2.2. Similarly, the vertices in Y_3 and Y_5 are of degree 1 in the subgraph $G_1[Y_3 \cup Y_4 \cup Y_5] \cap H$ (resp. $G_2[Y_3 \cup Y_4 \cup Y_5] \cap H$). Also, the vertices in Y_5 and Y_1 are of degree 1 in the subgraph $G_1[Y_5 \cup Y_6 \cup Y_1] \cap H$ (resp. $G_2[Y_5 \cup Y_6 \cup Y_1] \cap H$). This fact will be used in the proof of Lemma 2.9. □

For our future reference, we label the Hamilton cycles of the Hamilton cycle decomposition of G_1, obtained in Lemma 2.5, as H_i, $i \in A$, and we label the Hamilton cycles of the Hamilton cycle decomposition of G_2, obtained in Lemma 2.6, as H_i', $i \in D$.

Remark 2.8. Let $G_{1,1}$, $G_{1,2}$ and $G_{1,3}$ be the subgraphs of G_1 induced by $Y_1 \cup Y_2 \cup Y_3$, $Y_3 \cup Y_4 \cup Y_5$ and $Y_5 \cup Y_6 \cup Y_1$, respectively. From each Hamilton cycle H_i, $i \in A$, of G_1 that we have obtained in Lemma 2.5, we form a new graph $H_{i,1}$ as follows: let $G_{1,1}^i$, $G_{1,2}^i$ and $G_{1,3}^i$ denote the subgraphs $G_{1,1} \cap H_i$, $G_{1,2} \cap H_i$ and $G_{1,3} \cap H_i$ of G_1. $H_{i,1}$ is obtained by considering the disjoint copies of $G_{1,1}^i$, $G_{1,2}^i$ and $G_{1,3}^i$ and adding certain disjoint paths between $G_{1,1}^i$ and $G_{1,2}^i$, $G_{1,2}^i$ and $G_{1,3}^i$, and $G_{1,3}^i$ and $G_{1,1}^i$. From the construction of the Hamilton cycles H_i, $i \in A$, in G_1, the vertices in Y_1 and Y_3 (resp. Y_3 and Y_5, and Y_5 and Y_1) are of degree 1 in $G_{1,1}^i$ (resp. $G_{1,2}^i, G_{1,3}^i$); see Remark 2.7. Corresponding vertices of Y_3 in $G_{1,1}^i$ and Y_3 in $G_{1,2}^i$ are connected by vertex-disjoint paths of length, say, k_1 (all of whose internal vertices are new). Similarly, corresponding vertices of Y_5 (resp. Y_1) in $G_{1,2}^i$ (resp. $G_{1,3}^i$) and Y_5 (resp. Y_1) in $G_{1,3}^i$ (resp. $G_{1,1}^i$) are connected by vertex-disjoint paths of length, say, k_2 (resp. k_3) (all of whose internal vertices are new). Call the resulting graph $H_{i,1}$. Then $H_{i,1}$ is indeed a cycle. This is easy to see, because if we delete the internal vertices of the paths of length k_1, k_2 and k_3 (that we have used to connect the graphs $G_{1,1}^i$, $G_{1,2}^i$ and $G_{1,3}^i$) and identify the vertices of Y_3 (resp. Y_5, Y_1) in $G_{1,1}^i$ (resp. $G_{1,2}^i, G_{1,3}^i$) with their corresponding vertices in Y_3 (resp. Y_5, Y_1) of $G_{1,2}^i$ (resp. $G_{1,3}^i, G_{1,1}^i$), then what we get is H_i, the Hamilton cycle of G_1. We shall use this remark in the proof of Lemma 2.9. □

Lemma 2.9. *If $r \ge 5$ and $n \ge 3$ are odd and $m \ge 4$ is even, then $W_{2r} \times (K_m * \overline{K}_n)$ has a Hamilton cycle decomposition.*

Proof. First we obtain a digraph D' from W_{2r} as follows: replace the edge $x_i x_{i+r}$, $1 \le i \le r$, of W_{2r} by a symmetric pair of arcs and replace the edge $x_i x_{i+1}$, $1 \le i \le 2r$, of W_{2r} by two directed arcs from x_i to x_{i+1}, that is, with the same tail and head. Thus we have a 3-regular digraph, that is, $d^+ = 3 = d^-$. Now we decompose D' into three directed Hamilton cycles $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$, as follows:
$\vec{H}_1 = \{(x_{r+2i-1}, x_{2i-1}), (x_{2i-1}, x_{2i}) \mid 1 \le i \le r\}$, $\vec{H}_2 = \{(x_{2i-2}, x_{2i-1}), (x_{2i-1}, x_{r+2i-1}) \mid 1 \le i \le r\}$, $\vec{H}_3 = \{(x_i, x_{i+1}) \mid 1 \le i \le 2r\}$. Clearly, $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ are arc-disjoint directed Hamilton cycles of D'.

We colour the arcs (not a proper colouring) of $\overrightarrow{H_i}$, $i = 1, 2, 3$, with two colours a and b. Using this colouring and the Hamilton cycle decompositions of G_1 and G_2 described in Lemmas 2.5 and 2.6, we decompose $W_{2r} \times (K_m * \overline{K}_n)$ into Hamilton cycles.

First we colour arcs of $\overrightarrow{H_1}$ as follows: assign colour a to the arcs (x_{r+1}, x_1), (x_1, x_2), (x_{r+3}, x_3), (x_3, x_4), (x_{r+5}, x_5) and (x_5, x_6). Colour the remaining arcs as follows: colour the arc (x_{2i-1}, x_{2i}) $\big($resp. $(x_{r+2i-1}, x_{2i-1})\big)$, $4 \le i \le r$, with a (resp. b) if $2i - 1 \equiv 3 \pmod 4$ or with b (resp. a) if $2i - 1 \equiv 1 \pmod 4$.

Colour the arcs of $\overrightarrow{H_2}$ as follows: assign the colour a to the arcs (x_{2r}, x_1), (x_1, x_{r+1}), (x_2, x_3), (x_3, x_{r+3}), (x_4, x_5) and (x_5, x_{r+5}) and colour the remaining arcs as described below: colour (x_{2i-2}, x_{2i-1}) $\big($resp. $(x_{2i-1}, x_{r+2i-1})\big)$, $4 \le i \le r$, with a (resp. b) if $2i - 1 \equiv 3 \pmod 4$ or with b (resp. a) if $2i - 1 \equiv 1 \pmod 4$.

Colour the arcs of $\overrightarrow{H_3}$ as follows: assign the colour b to the arcs (x_{2r}, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5) and (x_5, x_6) and colour the remaining arcs (x_{2i-2}, x_{2i-1}) and (x_{2i-1}, x_{2i}), $4 \le i \le r$, with b if $2i-1 \equiv 3 \pmod 4$ or with a if $2i-1 \equiv 1 \pmod 4$.

Properties of the colouring

By the colouring described above, we can observe the following:

1. The arcs $(x_{2i-1}, x_{2i}) \in \overrightarrow{H_1}$ and $(x_{2i-1}, x_{2i}) \in \overrightarrow{H_3}$, $1 \le i \le r$, are assigned different colours.
2. The arcs $(x_{2i-2}, x_{2i-1}) \in \overrightarrow{H_2}$ and $(x_{2i-2}, x_{2i-1}) \in \overrightarrow{H_3}$, $1 \le i \le r$, are assigned different colours.
3. The arcs $(x_{r+2i-1}, x_{2i-1}) \in \overrightarrow{H_1}$ and $(x_{2i-1}, x_{r+2i-1}) \in \overrightarrow{H_2}$, $1 \le i \le r$, are assigned the same colour.
4. The arcs (x_{r+2i-1}, x_{2i-1}) and (x_{2i-1}, x_{2i}), $4 \le i \le r$, of $\overrightarrow{H_1}$ are assigned different colours.
5. The arcs (x_{2i-2}, x_{2i-1}) and (x_{2i-1}, x_{r+2i-1}), $4 \le i \le r$, of $\overrightarrow{H_2}$ are assigned different colours.
6. The arcs (x_{2i-2}, x_{2i-1}) and (x_{2i-1}, x_{2i}), $4 \le i \le r$, of $\overrightarrow{H_3}$ are assigned the colour b if $2i - 1 \equiv 3 \pmod 4$ or a if $2i - 1 \equiv 1 \pmod 4$.

Next we decompose $W_{2r} \times (K_m * \overline{K}_n)$ into Hamilton cycles using the colourings of the arcs of $\overrightarrow{H_i}$'s, $i = 1, 2, 3$, and the Hamilton cycle decompositions of G_1 and G_2.

From the construction of $\overrightarrow{H_1}$ in D', it is clear that the pairs of arcs $\{(x_{r+1}, x_1), (x_1, x_2)\}$, $\{(x_{r+3}, x_3), (x_3, x_4)\}$ and $\{(x_{r+5}, x_5), (x_5, x_6)\}$ describe three arc-disjoint directed paths of length 2 along $\overrightarrow{H_1}$; let these paths be $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$, respectively. $P_{1,2}$ occurs after $P_{1,1}$, and $P_{1,3}$ occurs after $P_{1,2}$, where we assume that x_{r+1} is the origin of $\overrightarrow{H_1}$. We shall use these $P_{1,j}$'s in the following construction.

Using the Hamilton cycles H_i, $i \in A$, of G_1 and $\overrightarrow{H_1}$ of D' we now construct a graph $H_{i,1}$ (which is actually a cycle) as follows: consider the directed Hamilton cycle $\overrightarrow{H_1}$ of D'. Arrange the X_i's of $W_{2r} \times (K_m * \overline{K}_n)$ according to the order of occurrence of the vertices x_i in $\overrightarrow{H_1}$; see Fig. 4.

We shall add some edges between the consecutive layers, in the above order, using some rule described below, and prove that the resulting graph $H_{i,1}$ is a Hamilton cycle of $W_{2r} \times (K_m * \overline{K}_n)$.

The corresponding edges of the Hamilton cycle H_i, $i \in A$, of G_1, joining the vertices of Y_1 and Y_2 are added between the layers X_{r+1} and X_1 with the preservation of the subscripts of the vertices, that is, if $y_k^1 y_l^2$ is an edge of H_i joining the vertex y_k^1 of Y_1 and the vertex y_l^2 of Y_2 in G_1, then the edge added between X_{r+1} and X_1 is $x_k^{r+1} x_l^1$. Similarly, the corresponding edges of the Hamilton cycle H_i, $i \in A$, of G_1 joining the vertices of Y_2 and Y_3 are added between the layers X_1 and X_2.

Similarly, add the corresponding edges of H_i, $i \in A$, of G_1 joining the vertices of Y_3 (resp. Y_5) and Y_4 (resp. Y_6) between the layers X_{r+3} (resp. X_{r+5}) and X_3 (resp. X_5) and also add the corresponding edges of H_i, $i \in A$, joining the vertices of Y_4 (resp. Y_6) and Y_5 (resp. Y_1) between the layers X_3 (resp. X_5) and X_4 (resp. X_6). Further, for all arcs (x_p, x_q) of $\overrightarrow{H_1}$, other than the arcs in $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$, add the edges of the 1-factor (of the subgraph of $W_{2r} \times (K_m * \overline{K}_n)$ induced by $X_p \cup X_q$) of distance i (resp. $mn-i$) from X_p to X_q if the colour assigned to the arc (x_p, x_q) of $\overrightarrow{H_1}$ is a (resp. b). Now we have fixed a 1-factor (of the subgraph of $W_{2r} \times (K_m * \overline{K}_n)$ induced by the consecutive layers) between two consecutive layers corresponding to the ends of the arcs of $\overrightarrow{H_1}$, other than the arcs in $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$. Call the resulting graph $H_{i,1}$. The subgraph of $H_{i,1}$ induced by $X_{r+1} \cup X_1 \cup X_2$ contains a disjoint union of paths of length 2 covering all of its vertices, wherein the vertices of X_{r+1} and X_2 are of degree 1; see Remark 2.7. Similarly, in the subgraph of $H_{i,1}$ induced by $X_{r+3} \cup X_3 \cup X_4$ (resp. $X_{r+5} \cup X_5 \cup X_6$) the vertices of X_{r+3} (resp. X_{r+5}) and X_4 (resp. X_6) are of degree 1; see Remark 2.7.

From the property (4) of the colouring described for the arcs of $\overrightarrow{H_1}$ and the fact that r is odd, it can be checked that among the arcs in the segment of $\overrightarrow{H_1}$ beginning from the terminus x_2 (resp. x_4, x_6) of $P_{1,1}$ (resp. $P_{1,2}$, $P_{1,3}$) and going to the origin x_{r+3} (resp. x_{r+5}, x_{r+1}) of $P_{1,2}$ (resp. $P_{1,3}$, $P_{1,1}$), the number of arcs receiving the colour a is the same as the number of arcs receiving the colour b. Consequently, from the kth vertex of X_2 (resp. X_4, X_6) to the kth vertex of X_{r+3} (resp. X_{r+5}, X_{r+1}) there is a path in $H_{i,1}$. Hence the graph $H_{i,1}$ is just like the construction described in the Remark 2.8 and hence it is a Hamilton

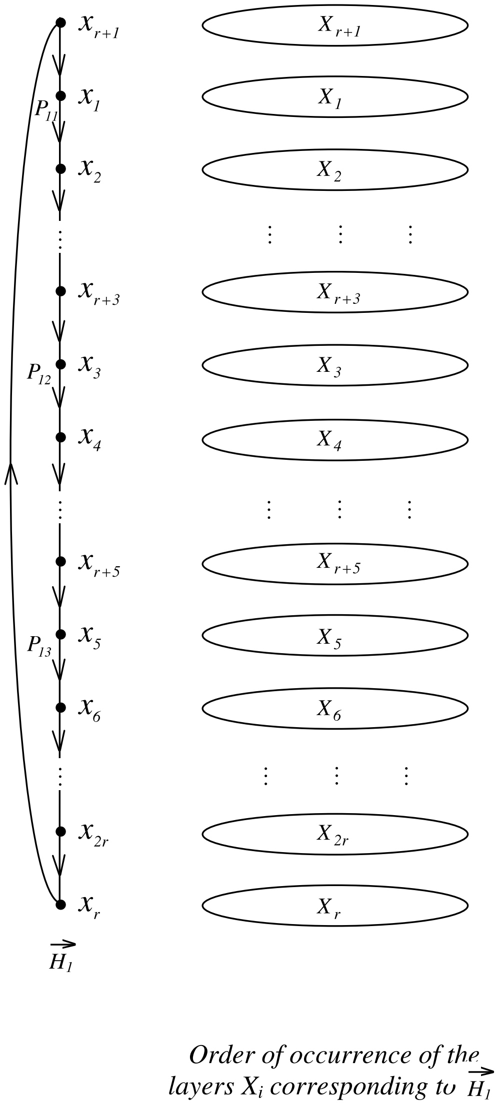


Fig. 4.

cycle. Thus $H_{i,1}$ is a Hamilton cycle of $W_{2r} \times (K_m * \overline{K}_n)$. Thus corresponding to the Hamilton cycles H_i of G_1 and $\overrightarrow{H_1}$ of D', we have obtained a Hamilton cycle $H_{i,1}$ of $W_{2r} \times (K_m * \overline{K}_n)$.

From the construction of $\overrightarrow{H_2}$, it is clear that the pairs of arcs $\{(x_{2r}, x_1), (x_1, x_{r+1})\}$, $\{(x_2, x_3), (x_3, x_{r+3})\}$ and $\{(x_4, x_5), (x_5, x_{r+5})\}$ describe three arc-disjoint directed paths of length 2 along $\overrightarrow{H_2}$; let these paths be $P_{2,1}$, $P_{2,2}$ and $P_{2,3}$, respectively. $P_{2,2}$ occurs after $P_{2,1}$, and $P_{2,3}$ occurs after $P_{2,2}$, where we assume that x_{2r} is the origin of $\overrightarrow{H_2}$. These paths $P_{2,j}$ are used in the following construction. We shall construct, like $H_{i,1}$, another graph $H_{i,2}$ (which is also a cycle) using the Hamilton cycle H_i, $i \in A$, of G_1 and the directed Hamilton cycle $\overrightarrow{H_2}$ of D'.

For the construction of $H_{i,2}$ consider the directed Hamilton cycle $\overrightarrow{H_2}$ of D'. Arrange the X_i's of $W_{2r} \times (K_m * \overline{K}_n)$ according to the order of occurrence of the vertices of x_i's in $\overrightarrow{H_2}$. The corresponding edges of the Hamilton cycle H_i, $i \in A$, of G_1, joining the vertices of Y_1 and Y_2 are added between the layers X_{2r} and X_1 with the preservation of the subscripts of the vertices. Also, the corresponding edges of the Hamilton cycle H_i, $i \in A$, of G_1 joining the vertices of Y_2 and Y_3 are added between the layers X_1 and X_{r+1}. Similarly, add edges between the layers X_2 and X_3, and X_3 and X_{r+3} (resp. X_4 and X_5, and X_5 and X_{r+5}) corresponding to the edges of H_i, $i \in A$, of G_1 between Y_3 and Y_4, and Y_4 and Y_5 (resp. Y_5 and Y_6, and Y_6 and Y_1). For the arcs (x_p, x_q) of $\overrightarrow{H_2}$ not in $P_{2,1}$, $P_{2,2}$ and $P_{2,3}$, add the edges of the 1-factor (of the subgraph of $W_{2r} \times (K_m * \overline{K}_n)$ induced by $X_p \cup X_q$) of distance i (resp. $mn - i$) from X_p to X_q if the colour assigned to the arc (x_p, x_q) is a (resp. b). Call the resulting graph $H_{i,2}$.

The subgraph of $H_{i,2}$ induced by $X_{2r} \cup X_1 \cup X_{r+1}$ (resp. $X_2 \cup X_3 \cup X_{r+3}, X_4 \cup X_5 \cup X_{r+5}$) contains only a disjoint union of paths of length 2 covering all of its vertices, wherein the vertices of $X_{2r} \cup X_{r+1}$ (resp. $X_2 \cup X_{r+3}, X_4 \cup X_{r+5}$) are of degree 1; see Remark 2.7. The proof of the fact that $H_{i,2}$ is a Hamilton cycle of $W_{2r} \times (K_m * \overline{K}_n)$ is similar to the proof for $H_{i,1}$. Thus corresponding to the Hamilton cycles H_i of G_1 and $\overrightarrow{H}_2$ of D', we have obtained a Hamilton cycle $H_{i,2}$ of $W_{2r} \times (K_m * \overline{K}_n)$.

From the construction of $\overrightarrow{H}_3$, it is clear that the arcs (x_{2r}, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5) and (x_5, x_6) describe a directed path $P_{3,1}$ of length 6 along $\overrightarrow{H}_3$, where we assume that x_{2r} is the origin of $\overrightarrow{H}_3$. Using the Hamilton cycles H'_i, $i \in D$, of G_2 and $\overrightarrow{H}_3$ of D' we shall construct a graph $H_{i,3}$ (which is actually a cycle) as follows: consider the directed Hamilton cycle $\overrightarrow{H}_3$ of D'. Arrange the X_i's of $W_{2r} \times (K_m * \overline{K}_n)$ according to the order of occurrence of the vertices x_i in $\overrightarrow{H}_3$. The corresponding edges of the Hamilton cycle H'_i, $i \in D$, of G_2 joining the vertices of Y_1 and Y_2 are added between the layers X_{2r} and X_1. Similarly, the corresponding edges of H'_i, $i \in D$, of G_2 joining the vertices of Y_2 (resp. Y_3, Y_4, Y_5, Y_6) and Y_3 (resp. Y_4, Y_5, Y_6, Y_1) are added between the layers X_1 (resp. X_2, X_3, X_4, X_5) and X_2 (resp. X_3, X_4, X_5, X_6).

Again, for all the arcs (x_p, x_q) of $\overrightarrow{H}_3$, other than the arcs in $P_{3,1}$, add the edges of the 1-factor (of the subgraph of $W_{2r} \times (K_m * \overline{K}_n)$ induced by $X_p \cup X_q$) of distance i (resp. $mn - i$) from X_p to X_q if the colour assigned to the arc (x_p, x_q) of $\overrightarrow{H}_3$ is b (resp. a). Call the resulting graph $H_{i,3}$. The subgraph of $H_{i,3}$ induced by $X_{2r} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \cup X_6$ contains only a disjoint union of paths covering all of its vertices, wherein the vertices of $X_{2r} \cup X_6$ are of degree 1.

From the property (6) of the colouring described for the arcs of $\overrightarrow{H}_3$ and the fact that r is odd, it follows that among the arcs from the terminus x_6 of $P_{3,1}$ to the origin x_{2r} of $P_{3,1}$ (taken along $\overrightarrow{H}_3$) the number of arcs receiving the colour a is same as the number of arcs receiving the colour b. Consequently, from the kth vertex of X_6 to the kth vertex of X_{2r} there is a path in $H_{i,3}$ (taken along the direction of $\overrightarrow{H}_3$). We can check that $H_{i,3}$ is a Hamilton cycle of $W_{2r} \times (K_m * \overline{K}_n)$. Let $F = \bigcup_{i=1}^{r} F_{mn/2}(X_{2i-1}, X_{r+2i-1})$. Clearly, F is a 1-factor of $W_{2r} \times (K_m * \overline{K}_n)$.

Next we shall show that $\{H_{i,1} \mid i \in A\}$, $\{H_{i,2} \mid i \in A\}$, $\{H_{i,3} \mid i \in D\}$ and F partitions the edge set of $W_{2r} \times (K_m * \overline{K}_n)$.

The arcs $(x_{r+2i-1}, x_{2i-1}) \in \overrightarrow{H}_1$ and $(x_{2i-1}, x_{r+2i-1}) \in \overrightarrow{H}_2$, $1 \le i \le r$, are assigned the same colour by property (3) and hence the Hamilton cycles in $\{H_{i,1}, H_{i,2} \mid i \in A\}$ contain the edges of the 1-factors of distances in $A \cup B$ from X_{2i-1} to X_{r+2i-1}, using the fact that $F_l(X_i, X_j) = F_{mn-l}(X_j, X_i)$, $A = \{mn - i \mid i \in B\}$ and $B = \{mn - i \mid i \in A\}$; F contains the edges of the 1-factor of distance in C $(= \{mn/2\})$ from X_{2i-1} to X_{r+2i-1}. Hence $\{H_{i,1}, H_{i,2} \mid i \in A\} \cup F$ contains all the edges of $W_{2r} \times (K_m * \overline{K}_n)$ joining the vertices of X_{2i-1} and X_{r+2i-1}.

The arcs $(x_{2i-1}, x_{2i}) \in \overrightarrow{H}_1$ and $(x_{2i-1}, x_{2i}) \in \overrightarrow{H}_3$, $1 \le i \le r$, are assigned different colours a and b. If $(x_{2i-1}, x_{2i}) \in \overrightarrow{H}_1$ is assigned the colour a (resp. b) and $(x_{2i-1}, x_{2i}) \in \overrightarrow{H}_3$ is assigned the colour b (resp. a), then the Hamilton cycles in $\{H_{i,1} \mid i \in A\} \cup \{H_{i,3} \mid i \in D\}$ contain all the edges of the 1-factors of distances in $E = A \cup D$, where $D = B \cup C$ (resp. $E = B \cup (A \cup C)$, where $A \cup C = \{mn - i \mid i \in D\}$) from X_{2i-1} to X_{2i}. Hence the Hamilton cycles in $\{H_{i,1} \mid i \in A\} \cup \{H_{i,3} \mid i \in D\}$ contain all the edges of $W_{2r} \times (K_m * \overline{K}_n)$ joining the vertices of X_{2i-1} and X_{2i}.

Similarly, the arcs $(x_{2i-2}, x_{2i-1}) \in \overrightarrow{H}_2$ and $(x_{2i-2}, x_{2i-1}) \in \overrightarrow{H}_3$, $1 \le i \le r$, are assigned different colours a and b. As above, we can check that the Hamilton cycles in $\{H_{i,2} \mid i \in A\} \cup \{H_{i,3} \mid i \in D\}$ contain all the edges of $W_{2r} \times (K_m * \overline{K}_n)$ joining the vertices of X_{2i-2} and X_{2i-1}.

Thus $\{H_{i,1}, H_{i,2} \mid i \in A\} \cup \{H_{i,3} \mid i \in D\} \cup F$ is a Hamilton cycle decomposition of $W_{2r} \times (K_m * \overline{K}_n)$. This completes the proof. □

We use the following results in the proof of Theorem 1.1.

Lemma 2.10 ([6]). *If both G_1 and G_2 have Hamilton cycle decompositions and at least one of G_1 and G_2 is of odd order, then $G_1 \times G_2$ is Hamilton cycle decomposable.* □

Lemma 2.11 ([10]). *If $r \ge 3$, then $C_r * \overline{K}_s$ is Hamilton cycle decomposable.* □

Lemma 2.12 ([11]). *Let $r \ge 3$ be odd. Then $K_{r,r}$ can be decomposed into Hamilton cycles and a $W_{2r} (\cong X(2r; \{1, r\}))$, that is, $K_{r,r} = C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r} \oplus W_{2r}$.* □

Lemma 2.13 ([11]). *If $m, n \ge 2$, then $C_{2n} \times K_{2m}$ is Hamilton cycle decomposable.* □

Theorem 2.14 ([11]). *If $m \ge 3$, then $K_{r,r} \times K_m$ is Hamilton cycle decomposable.* □

Lemma 2.15. *If $m \ge 4$ is even and $n, r \ge 3$ are odd, then $W_{2r} \times (K_m * \overline{K}_n)$ is Hamilton cycle decomposable.*

Proof. If $r = 3$, then the conclusion follows from Lemmas 2.4–2.6. If $r \ge 5$, then it follows from Lemma 2.9. □

Proof of Theorem 1.1. If $n = 1$, then the result follows from Theorem 2.14. Hence we may assume that $n \ge 2$. We prove this theorem in two cases.

Case 1. r is even.

As $K_{r,r}$ is Hamilton cycle decomposable, $K_{r,r} = C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r}$ and hence

$$\begin{aligned} K_{r,r} \times (K_m * \overline{K}_n) &= \left(C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r}\right) \times (K_m * \overline{K}_n) \\ &= C_{2r} \times (K_m * \overline{K}_n) \oplus C_{2r} \times (K_m * \overline{K}_n) \oplus \cdots \oplus C_{2r} \times (K_m * \overline{K}_n), \end{aligned}$$

since the tensor product is distributive over edge-disjoint subgraphs. It is enough to prove that $C_{2r} \times (K_m * \overline{K}_n)$ is Hamilton cycle decomposable.

Clearly,

$$\begin{aligned} C_{2r} \times (K_m * \overline{K}_n) &\cong \left(C_{2r} \times K_m\right) * \overline{K}_n \\ &= \left(C_{2rm} \oplus C_{2rm} \oplus \cdots \oplus C_{2rm}\right) * \overline{K}_n, \quad \text{by Lemma 2.10 or Lemma 2.13.} \\ &= C_{2rm} * \overline{K}_n \oplus C_{2rm} * \overline{K}_n \oplus \cdots \oplus C_{2rm} * \overline{K}_n. \end{aligned}$$

But $C_{2rm} * \overline{K}_n$ is Hamilton cycle decomposable, by Lemma 2.11. Thus we have a Hamilton cycle decomposition of $K_{r,r} \times (K_m * \overline{K}_n)$.

Case 2. r is odd.

We complete the proof in two subcases.

Subcase 2.1. m is odd.

$$\begin{aligned} K_{r,r} \times (K_m * \overline{K}_n) &\cong (K_{r,r} \times K_m) * \overline{K}_n \\ &= \left(C_{2rm} \oplus C_{2rm} \oplus \cdots \oplus C_{2rm}\right) * \overline{K}_n, \quad \text{by Theorem 2.14} \\ &= C_{2rm} * \overline{K}_n \oplus C_{2rm} * \overline{K}_n \oplus \cdots \oplus C_{2rm} * \overline{K}_n. \end{aligned}$$

As $C_{2rm} * \overline{K}_n$ is Hamilton cycle decomposable, by Lemma 2.11, we have a Hamilton cycle decomposition of $K_{r,r} \times (K_m * \overline{K}_n)$.

Subcase 2.2. m is even.

$$K_{r,r} = \underbrace{C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r}}_{(r-3)/2\text{times}} \oplus W_{2r}, \quad \text{by Lemma 2.12.}$$

Consequently,

$$\begin{aligned} K_{r,r} \times (K_m * \overline{K}_n) &= \left(C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r} \oplus W_{2r}\right) \times (K_m * \overline{K}_n) \\ &= C_{2r} \times (K_m * \overline{K}_n) \oplus C_{2r} \times (K_m * \overline{K}_n) \oplus \cdots \oplus C_{2r} \times (K_m * \overline{K}_n) \oplus W_{2r} \times (K_m * \overline{K}_n). \end{aligned}$$

It is enough to prove that $C_{2r} \times (K_m * \overline{K}_n)$ and $W_{2r} \times (K_m * \overline{K}_n)$ are Hamilton cycle decomposable.

Clearly,

$$\begin{aligned} C_{2r} \times (K_m * \overline{K}_n) &\cong \left(C_{2r} \times K_m\right) * \overline{K}_n \\ &= \left(C_{2rm} \oplus C_{2rm} \oplus \cdots \oplus C_{2rm}\right) * \overline{K}_n, \quad \text{by Lemma 2.13,} \\ &= C_{2rm} * \overline{K}_n \oplus C_{2rm} * \overline{K}_n \oplus \cdots \oplus C_{2rm} * \overline{K}_n. \end{aligned}$$

But $C_{2rm} * \overline{K}_n$ is Hamilton cycle decomposable, by Lemma 2.11. Hence $C_{2r} \times (K_m * \overline{K}_n)$ is Hamilton cycle decomposable. As $m \geq 4$ is even and $r \geq 3$ is odd, the existence of a Hamilton cycle decomposition of $W_{2r} \times (K_m * \overline{K}_n)$ follows from Lemma 2.3 or Lemma 2.15 according to whether n is even or odd, respectively. This completes the proof. □

Acknowledgements

The authors would like to thank the Department of Science and Technology, Government of India, New Delhi, for partial financial assistance through the Project Grant No. SR/S4/MS:217/03.

References

[1] B. Alspach, J.-C. Bermond, D. Sotteau, in: G. Hahn, et al. (Eds.), Decompositions into Cycles I: Hamilton Decompositions: in Cycles and Rays, Kluwer Academic Press, Dordrecht, 1990, pp. 9–18.
[2] R. Balakrishnan, J.-C. Bermond, P. Paulraja, M.-L. Yu, On Hamilton cycle decompositions of the tensor product of complete graphs, Discrete Math. 268 (2003) 49–58.
[3] R. Balakrishnan, P. Paulraja, Hamilton cycles in tensor product of graphs, Discrete Math. 186 (1998) 1–13.
[4] R. Balakrishnan, K. Ranganathan, A Text Book of Graph Theory, Springer-Verlag, New York, 2000.
[5] Z. Baranyai, G. Szasz, Hamiltonian decompositions of lexicographic product, J. Combin. Theory Ser. B 31 (1981) 253–261.
[6] J.-C. Bermond, Hamiltonian decompositions of graphs, digraphs and hypergraphs, Ann. Discrete Math. 3 (1978) 21–28.
[7] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Elsevier, New York, 1976.
[8] J. Bosak, Decompositions of Graphs, Kluwer Academic Press, Dordrecht, 1990.

[9] P. Jha, Hamilton decompositions of product of cycles, Indian J. Pure Appl. Math. 23 (1992) 723–729.

[10] R. Laskar, Decomposition of some composite graphs into Hamiltonian cycles, in: Proc. 5th Hungarian Coll. Keszthely 1976, North Holland, 1978, pp. 705–716.

[11] R.S. Manikandan, P. Paulraja, Hamiltonian decompositions of the tensor product of a complete graph and a complete bipartite graph, Ars Combin. 80 (2006) 33–44.

[12] R.S. Manikandan, P. Paulraja, Hamilton cycle decompositions of the tensor product of complete multipartite graphs, Discrete Math. 308 (2008) 3586–3606.

[13] L. Ng, Hamiltonian Decomposition of Lexicographic Products of Digraphs, J. Combin. Theory (ser. B) 73 (1998) 119–129.

[14] R. Stong, Hamilton decompositions of cartesian product of graphs, Discrete Math. 90 (1991) 169–190.